

TRANSCIENCE OF EDGE-REINFORCED RANDOM WALK

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ABSTRACT. We show transience of the edge-reinforced random walk for small reinforcement in dimension $d \geq 3$. The argument adapts the proof of quasi-diffusive behaviour of the SuSy hyperbolic model for fixed conductances by Disertori, Spencer and Zirnbauer [7], using the representation of edge-reinforced random walk as a mixture of vertex-reinforced jump processes (VRJP) with independent gamma conductances, and the interpretation of the limit law of VRJP as a supersymmetric (SuSy) hyperbolic sigma model developed by Sabot and Tarrès in [9].

1. INTRODUCTION

1.1. Setting and main result. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $G = (V, E, \sim)$ be a nonoriented connected locally finite graph without loops. Let $(a_e)_{e \in E}$ and $(w_e)_{e \in E}$ two sequences of positive weights associated to each edge $e \in E$.

Let $(X_n)_{n \in \mathbb{N}}$ be a random process that takes values in V , and let $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$ be the filtration of its past. For any $e \in E$, $n \in \mathbb{N} \cup \{\infty\}$, let

$$(1.1) \quad Z_n(e) = a_e + \sum_{k=1}^n \mathbb{1}_{\{X_{k-1}, X_k\} = e}$$

be the number of crosses of e up to time n plus the initial weight a_e . Then $(X_n)_{n \in \mathbb{N}}$ is called Edge Reinforced Random Walk (ERRW) with starting point $i_0 \in V$ and weights $(a_e)_{e \in E}$, if $X_0 = i_0$ and, for all $n \in \mathbb{N}$,

$$(1.2) \quad \mathbb{P}(X_{n+1} = j \mid \mathcal{F}_n) = \mathbb{1}_{\{j \sim X_n\}} \frac{Z_n(\{X_n, j\})}{\sum_{k \sim X_n} Z_n(\{X_n, k\})}.$$

On the other hand, let $(Y_t)_{t \geq 0}$ be a continuous-time process on V , starting at time 0 at some vertex $i_0 \in V$. Then $(Y_t)_{t \geq 0}$ is called a Vertex-Reinforced Jump Process (VRJP) with starting point i_0 and weights $(w_e)_{e \in E}$ if $Y_0 = i_0$ and if $Y_t = i$ then, conditionally on $(Y_s, s \leq t)$, the process jumps to a neighbour j of i at rate $w_{\{i,j\}} L_j(t)$, where

$$L_j(t) := 1 + \int_0^t \mathbb{1}_{\{Y_s = j\}} ds.$$

The Edge Reinforced Random Walk was introduced in 1986 by Diaconis [2]; the Vertex-Reinforced Jump Process was proposed by Werner in 2000, and initially studied by Davis and Volkov [3, 4]; for more details on these models and related questions, see [9] for instance.

The aim of this paper is to prove transience of the edge-reinforced random walk (ERRW) for large $a_e > 0$, i.e. small reinforcement.

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Theorem 1. *On \mathbb{Z}^d , $d \geq 3$, there exists $\alpha_c(d) > 0$ such that, if $a_e > \alpha_c(d)$ for all $e \in E$, then the ERRW with weights $(a_e)_{e \in E}$ is transient a.s.*

Recall that almost-sure positive recurrence of ERRW and VRJP for large reinforcement (i.e. if $a_e < \tilde{\alpha}_c$, resp. $w_e < \tilde{w}_c$ for all $e \in E$, for some $\tilde{\alpha}_c, \tilde{w}_c > 0$), and transience of VRJP for small reinforcement (i.e. if $w_e > w_c$ for some $w_c > 0$) were proved by Sabot and Tarrès in [9], using localisation/delocalisation results of Disertori, Spencer and Zirnbauer [6, 7]. Another proof of recurrence of the ERRW and VRJP was proposed, shortly afterwards, by Angel, Crawford and Kozma [1].

The proof of Theorem 1 follows from estimates on the fluctuation of a field (U_i) associated to the limiting behaviour of the reinforced-random walk. Let us first recall two earlier results from Sabot and Tarrès [9].

Theorem 2 (Sabot and Tarrès [9]). *On any locally finite graph G , the ERRW $(X_n)_{n \geq 0}$ is equal in law to the discrete time process associated with a VRJP in random independent conductances $W_e \sim \text{Gamma}(a_e, 1)$.*

The next result concerns VRJP $(Y_t)_{t \geq 0}$ on a finite graph G , with $|V| = N$, given fixed weights $(w_e)_{e \in E}$; let $\mathbb{P}_{i_0}^{VRJP}$ be its law, starting from $i_0 \in V$.

Proposition 1 (Sabot and Tarrès [9]). *Suppose that G is finite and set $N = |V|$. For all $i \in V$, the following limits exist $\mathbb{P}_{i_0}^{VRJP}$ a.s.*

$$U_i = \lim_{t \rightarrow \infty} (\log L_i(t) - \log L_{i_0}(t)).$$

Theorem 3 (Sabot and Tarrès [9]). **(i)** *Under $\mathbb{P}_{i_0}^{VRJP}$, $(U_i)_{i \in V}$ has the following distribution on*

$$\mathcal{H}_0 = \{(u_i) \in \mathbb{R}^V : u_{i_0} = 0\}$$

$$(1.3) \quad d\rho_{w, \Lambda_n}(u) = \frac{1}{(2\pi)^{(N-1)/2}} e^{-\sum_{j \in V} u_j} e^{-H(w, u)} \sqrt{D[m(w, u)]} \prod_{j \in V \setminus \{i_0\}} du_j$$

where

$$H(w, u) = 2 \sum_{\{i, j\} \in E} w_{i, j} (\cosh(u_i - u_j) - 1)$$

and $D[m(w, u)]$ is any diagonal minor of the $N \times N$ matrix $m(w, u)$ with coefficients

$$m_{i, j}(w, u) = \begin{cases} w_{i, j} e^{u_i + u_j} & \text{if } i \neq j \\ -\sum_{k \in V} w_{i, k} e^{u_i + u_k} & \text{if } i = j \end{cases}$$

(ii) *Let C be the following positive continuous additive functional of X :*

$$C(s) = \sum_{i \in V} L_i^2(s) - 1,$$

and let

$$Z_t = Y_{C^{-1}(t)}.$$

Then, conditionally on $(U_i)_{i \in V}$, Z_t is a Markov jump process starting from i_0 , with jump rate from i to j

$$\frac{1}{2} w_{i, j} e^{U_j - U_i}.$$

In particular, the discrete time process associated with $(Y_s)_{s \geq 0}$ is a mixture of reversible Markov chains with conductances $w_{i,j}e^{U_i+U_j}$.

Remark 1. The diagonal minors of the matrix $m(w, u)$ are all equal since the sums on any line or column of the coefficients of the matrix are null. By the matrix-tree theorem, if we let \mathcal{T} be the set of spanning trees of (V, E, \sim) , then $D[m(w, u)] = \sum_{T \in \mathcal{T}} \prod_{\{i,j\} \in T} w_{i,j} e^{u_i+u_j}$.

Notation and convention. In the following we fix $d \geq 3$.

A sequence $\sigma = (x_0, \dots, x_n)$ is a path from x to y in \mathbb{Z}^d if $x_0 = x$, $x_n = y$ and $x_{i+1} \sim x_i$ for all $i = 1, \dots, n$.

Let

$$\Lambda_n = \{i \in \mathbb{Z}^d, |i|_\infty \leq n\}$$

be the ball centred at 0 with radius n , and let

$$\partial\Lambda_n = \{i \in \mathbb{Z}^d, |i|_\infty = n\}$$

be its boundary. We denote by E the set of edges in \mathbb{Z}^d and by E_n the set of edges contained in the hypercube Λ_n . We denote by \tilde{E}_n the associated set of directed edges.

Let $(a_{ij})_{i,j \in \mathbb{Z}^d, i \sim j}$ be the family of initial positive weights for the ERRW, and let

$$a := \inf_{e \in E} a_e.$$

In the proofs, we will denote by $\text{Cst}(a_1, a_2, \dots, a_p)$ a positive constant depending only on a_1, a_2, \dots, a_p , and by Cst a universal positive constant.

Let \mathbb{P}_0 (resp. $\mathbb{P}_0^{\Lambda_n}$) be the law of the ERRW on \mathbb{Z}^d (resp. on Λ_n). Theorem 3 ensures that on every finite volume Λ_n the ERRW is a mixture of reversible Markov chains with random conductances $(W_e^U)_{e \in E_n}$ where $W_{ij}^U = W_{ij}e^{U_i+U_j}$ and the law for the random variables (W, U) has joint distribution on $\mathbb{R}_+^{\tilde{E}_n} \times \mathcal{H}_0$ given by

$$d\rho_{\Lambda_n}(w, u) = d\rho_{w, \Lambda_n}(u) \prod_{e \in E_n} \frac{e^{-w_e} w_e^{a_e-1}}{\Gamma(a_e)} dw_e,$$

where $d\rho_{w, \Lambda_n}(u)$ was defined in (1.3). Let $\mathbb{E}_0^{\Lambda_n}$ be the average with respect to the joint law for (W, U) (mixing measure). Note that we cannot define this average on an infinite volume, since we do not know if the limiting measure exists. Denote by $\langle \cdot \rangle_{\Lambda_n}$ the corresponding marginal in U .

Warning. We use a capital letter $W, U \dots$ to denote a random variable and a smallcase letter w, u, \dots to denote a particular realization of the variable. The same is true for any function of such variables. In some cases though we do not state the argument explicitly, to avoid heavy notations. It should become clear from the context when the corresponding argument has to be regarded as a random variable.

The proof of Theorem 1 will follow from the following result.

Theorem 4. Fix $d \geq 3$. For all $m > 0$, there exists $a_c(m, d) > 0$ such that, if $a \geq a_c(m, d)$ then, for all $n \in \mathbb{N}$, $x, y \in \Lambda_n$,

$$\langle \cosh^m(U_x - U_y) \rangle_{\Lambda_n} \leq 2.$$

The proof of Theorem 4 is the purpose of the rest of this paper, and adapts the argument of Disertori, Spencer and Zirnbauer [7], which implied in particular transience of VRJP with large conductances (w_e). Let us first show how Theorem 4 implies Theorem 1.

Proof of Theorem 1. Given $(w, u) \in \mathbb{R}_+^E \times \mathcal{H}_0$, denote by $P_0^{w^u}$ the law of the Markov chain in conductances $w_{i,j}^u = w_{ij}e^{u_i+u_j}$ starting from 0. Let $H_{\partial\Lambda_n}$ be the first hitting time of the boundary $\partial\Lambda_n$ and \tilde{H}_0 be the first return time to the point 0.

We seek to estimate $\mathbb{P}_0(H_{\partial\Lambda_n} < \tilde{H}_0)$ the probability that the ERRW hits the boundary of Λ_n before coming back to 0. Since this event depends only on the history of the walk inside the hypercube Λ_n , we have

$$(1.4) \quad \mathbb{P}_0(H_{\partial\Lambda_n} < \tilde{H}_0) = \mathbb{P}_0^{\Lambda_n}(H_{\partial\Lambda_n} < \tilde{H}_0) = \mathbb{E}_0^{\Lambda_n} \left[P_0^{W^U}(H_{\partial\Lambda_n} < \tilde{H}_0) \right]$$

where in the last equality we used the last part of Theorem 3. Now $P_0^{w^u}(H_{\partial\Lambda_n} < \tilde{H}_0)$ is the probability for a Markov chain inside Λ_n with conductances w^u to hit the boundary before coming back to 0. This probability is related, by (1.5), to the effective resistance between 0 and $\partial\Lambda_n$ (see for instance in Chapter 2 of [8])

$$R(0, \partial\Lambda_n, w^u) = \inf_{\theta: \tilde{E}_n \rightarrow \mathbb{R}} \frac{1}{2} \sum_{e \in \tilde{E}_n} \frac{\theta(e)^2}{w_e^u}$$

where the infimum is taken on unit flows θ from 0 to $\partial\Lambda_n$: θ is defined on the set of directed edges \tilde{E}_n and must satisfy $\theta((i, j)) = -\theta((j, i))$ and $\sum_{j \sim v} \theta((v, j)) = \delta_{v,0}$ for all $v \in \Lambda_n$. Denote by $R(0, \partial\Lambda_n)$ the effective resistance between 0 and $\partial\Lambda_n$ for conductances 1.

Classically we have

$$(1.5) \quad w_0^u R(0, \partial\Lambda_n, w^u) = \frac{1}{P_0^{w^u}(H_{\partial\Lambda_n} < \tilde{H}_0)}$$

with $w_0^u = \sum_{j \sim 0} w_{0,j}^u$. Using (1.4) and Jensen's inequality

$$(1.6) \quad \begin{aligned} \frac{1}{\mathbb{P}_0(H_{\partial\Lambda_n} < \tilde{H}_0)} &= \frac{1}{\mathbb{E}_0^{\Lambda_n} \left[P_0^{W^U}(H_{\partial\Lambda_n} < \tilde{H}_0) \right]} \\ &\leq \mathbb{E}_0^{\Lambda_n} \left[\frac{1}{P_0^{W^U}(H_{\partial\Lambda_n} < \tilde{H}_0)} \right] = \mathbb{E}_0^{\Lambda_n} [W_0^U R(0, \partial\Lambda_n, W^U)]. \end{aligned}$$

We will show below that, if $\min_{e \in E} a_e = a > \max\{a_c(3, d), 3\}$, then

$$(1.7) \quad \mathbb{E}_0^{\Lambda_n} [W_0^U R(0, \partial\Lambda_n, W^U)] \leq \text{Cst}(a, d) R(0, \partial\Lambda_n)$$

This will enable us to conclude: since $\limsup R(0, \partial\Lambda_n) < \infty$ for all $d \geq 3$, (1.6) and (1.7) imply that $\mathbb{P}_0(\tilde{H}_0 = \infty) > 0$. \square

Proof of (1.7). Let θ be the unit flow from 0 to $\partial\Lambda_n$ which minimizes the L^2 norm. Then

$$R(0, \partial\Lambda_n, w_u) \leq \frac{1}{2} \sum_{(i,j) \in \tilde{E}_n} \frac{1}{w_{i,j}^u} \theta^2(i, j),$$

and

$$R(0, \partial\Lambda_n) = \frac{1}{2} \sum_{(i,j) \in \tilde{E}_n} \theta^2(i, j).$$

Now, if $a > \max\{\tilde{a}_c(3, d), 3\}$ then, using Theorem 4,

$$\begin{aligned} \mathbb{E}_0^{\Lambda_n} \left[\frac{W_0^U}{W_{i,j}^U} \right] &= \sum_{l \sim 0} \mathbb{E}_0^{\Lambda_n} \left[\frac{W_{0,l}}{W_{i,j}} e^{U_0 - U_i} e^{U_l - U_j} \right] \leq \sum_{l \sim 0} \left(\mathbb{E}_0^{\Lambda_n} \left[\frac{W_{0,l}^3}{W_{i,j}^3} \right] \mathbb{E}_0^{\Lambda_n} [e^{3(U_0 - U_i)}] \mathbb{E}_0^{\Lambda_n} [e^{3(U_l - U_j)}] \right)^{1/3} \\ &= \sum_{l \sim 0} (\mathbb{E}_0^{\Lambda_n} [W_{0,l}^3] \mathbb{E}_0^{\Lambda_n} [W_{i,j}^{-3}] \langle e^{3(U_0 - U_i)} \rangle_{\Lambda_n} \langle e^{3(U_l - U_j)} \rangle_{\Lambda_n})^{1/3} \leq \text{Cst}(a, d), \end{aligned}$$

where we also used the fact that $W_{i,j}$ are independent Gamma distributed random variables. \square

1.2. Organization of the paper. Section 2 introduces to the main lines of the proof, the geometric objects that are needed (diamonds and deformed diamonds), and state some Ward identities. Section 3 provides some estimates on the probability of existence of good points i (from which there are no large deviations on B_{ij} at small scales) in certain boxes. Section 4 contains the main inductive argument. The Ward identities (Lemmas 1 and 4) are shown in Section 5, and the estimate on the effective conductance given in Proposition 3 is proved in Section 6.

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2. INTRODUCTION TO THE PROOF

2.1. Marginals of U and a first Ward identity. Let us now fix $n \in \mathbb{N}$, and let $\Lambda = \Lambda_n$, $E = E_n$, for simplicity.

A key step in our proof is to study the law of U after integration over the conductances W_e , in other words to focus on the marginal $\langle \cdot \rangle_\Lambda$. Note that it is one of several possible approaches to the question of recurrence/transience for ERRW. Indeed, one could instead focus on the analysis of the law of the random conductances $W_{ij} e^{U_i + U_j}$ given by Coppersmith-Diaconis formula, or directly study ERRW from its tree of discovery using the existence of a limiting environment, as done in [1], or possibly conclude from a.s. results conditionally on the conductances W_e , $e \in E$.

In fact we add a Gaussian Free Field variable S with conductances W_{ij}^U before integrating over W , as in the first step of [7], since the corresponding joint law again has more transparent symmetries and is better suited for the subsequent analysis. More precisely, let

$$B_{xy} := \cosh(U_x - U_y) + \frac{1}{2} e^{U_x + U_y} (S_x - S_y)^2$$

for all $x, y \in V$ (not necessarily neighbours).

Then (W, U, S) has the following distribution on $\mathbb{R}_+^E \times \mathcal{H}_0 \times \mathcal{H}_0$

$$(2.1) \quad \frac{1}{(2\pi)^{(N-1)/2}} e^{-\sum_{j \in \Lambda} u_j} e^{-\sum_{i,j \sim i} W_{ij}(B_{ij}-1)} D[m(W, u)] \prod_{k \neq i_0} du_k ds_k \prod_{e \in E} \frac{e^{-w_e} w_e^{a_e-1}}{\Gamma(a_e)} dw_e$$

The marginal of this law in (U, S) after integration over W , which we still call $\langle \cdot \rangle_\Lambda$ or $\langle \cdot \rangle$ by a slight abuse of notation, is given in the following Proposition 2.

Proposition 2. *The joint variables (U, S) have density on $\mathcal{H}_0 \times \mathcal{H}_0$*

$$(2.2) \quad \mu_{a,V}(u, s) = \frac{1}{(2\pi)^{(N-1)/2}} \left[\prod_e \frac{1}{B_e^{a_e}} \right] e^{-\sum_{j \in \Lambda} u_j} D[M_V(u, s)],$$

where $D[M_V(u, s)]$ is any diagonal minor of the $N \times N$ matrix $M_V(u, s)$ defined by

$$M_{i,j} = \begin{cases} c_{i,j}, & i \neq j \\ -\sum_k c_{i,k}, & i = j \end{cases}$$

and

$$c_{i,j} := \frac{a_{i,j} e^{u_i+u_j}}{B_{i,j}}, \quad \forall i \sim j$$

Note that we slightly abuse notation here, in that $B_{i,j}$ is considered alternatively as function of (u, s) and (U, S) .

Proof. Integrating the density in (2.1) with respect to the random independent conductances $W_e \sim \text{Gamma}(a_e, 1)$ we obtain

$$\begin{aligned} & \frac{e^{-\sum_j u_j}}{(2\pi)^{(N-1)/2}} \int \prod_e \frac{e^{-w_e} w_e^{a_e-1} dw_e}{\Gamma(a_e)} e^{-\sum_{j \sim i} w_{ij}(B_{ij}-1)} D(w, u) \\ &= \frac{e^{-\sum_j u_j}}{(2\pi)^{(N-1)/2}} \sum_T \prod_{i \sim j \in T} e^{u_i+u_j} \left[\prod_{e \notin T} I_e(a_e) \right] \left[\prod_{e \in T} I_e(a_e + 1) \right] \\ &= \left[\prod_e \frac{1}{B_e^{a_e}} \right] \frac{e^{-\sum_j u_j}}{(2\pi)^{(N-1)/2}} \sum_T \prod_{i \sim j \in T} \frac{a_{i,j} e^{u_i+u_j}}{B_{i,j}} = \mu_{a,V}(u, s) \end{aligned}$$

where in the second line we expand the determinant as a sum over spanning trees and

$$I_e(a_e) = \int \frac{dw_e}{\Gamma(a_e)} e^{-w_e B_e} w_e^{a_e-1} = \frac{1}{B_e^{a_e}}, \quad I_e(a_e + 1) = \frac{a_e}{B_e^{a_e+1}}.$$

□

In the proof of delocalization in [7], the first step was a set of relations (Ward identities) generated by internal symmetries. Their analogue is stated in Lemma 1 and proved in Section 5, and involves a term of effective resistance $D_{x,y}$ depending on the variable U , defined as follows.

Definition 1. *Let $D_{x,y}$ be the effective resistance between x and y for the conductances*

$$c_{i,j}^{x,y} := c_{i,j} B_{x,y} e^{-u_x - u_y} = \frac{a_{i,j} e^{u_i+u_j-u_x-u_y} B_{x,y}}{B_{i,j}}.$$

Lemma 1. *For all $m \leq a/4$ then, for all $x, y \in \Lambda$ (not necessarily neighbours),*

$$\langle B_{x,y}^m (1 - m D_{x,y}) \rangle = 1.$$

One of the consequences of Lemma 1 is that, if the effective resistance D_{xy} is small, then we can deduce a good bound on $B_{x,y}^m$. However, there is positive probability that this resistance D_{xy} might be large, so that it would be more useful to show an inequality restricted to the event that D_{xy} is small.

Although it is not possible to derive such an identity directly, we can define events $\bar{\chi}_{xy}$ on which such identities hold, and so that D_{xy} is small on $\bar{\chi}_{xy}$. These events will depend on geometric objects defined in the next Section 2.2, called diamonds and deformed diamonds.

2.2. Diamonds and deformed diamonds.

Definition 2. *Let $l \neq 0$ be a vector in \mathbb{R}^d and $x \in \mathbb{Z}^d$. We denote by C_x^l the cone with base x , direction l and angle $\pi/4$*

$$\begin{aligned} C_x^l &= \{z \in \mathbb{R}^d, \quad \angle(xz, l) \leq \frac{\pi}{4}\} \\ &= \{z \in \mathbb{R}^d, \quad (z - x) \cdot l \geq \frac{\sqrt{2}}{2} |z - x| |l|\} \end{aligned}$$

where $\angle(xz, l)$ is the angle between the vector xz and the vector l .

If x and y are in \mathbb{Z}^d , we call *Diamond* the set

$$(C_x^{y-x} \cap C_y^{x-y}) \cap \mathbb{Z}^d.$$

to which we add a few points close to x and close to y so that the set is connected in \mathbb{Z}^d . We denote this set by $R_{x,y}$.

Remark 2. *By the expression "a few points" above we mean that we add some extra points to $(C_x^{y-x} \cap C_y^{x-y}) \cap \mathbb{Z}^d$ at a bounded distance from x, y so that the resulting set becomes connected in the lattice. The distance at which the points can be added is bounded by a constant depending only on the dimension. Of course, all the estimates below will be independent of the choice of these points. We will repeat this operation several times hereafter, without extra explanation.*

In the course of the inductive argument, some deformed diamonds appear, they are formed of the intersection of two cones with smaller angles than for diamonds. For $l \in \mathbb{R}^d$, $l \neq 0$, and $x \in \mathbb{R}^d$ we set

$$\tilde{C}_x^l = \{z \in \mathbb{R}^d, \quad \angle(xz, l) \leq \frac{\pi}{16}\}.$$

Definition 3. *A deformed diamond is a set of the following form*

$$(\tilde{C}_x^l \cap \tilde{C}_y^{x-y}) \cap \mathbb{Z}^d,$$

(plus a few points close to x and to y so that the set is connected in \mathbb{Z}^d , see Remark 2) where $x \in \mathbb{Z}^d$, $l \in \mathbb{R}^d$, $l \neq 0$ and $y \in \mathbb{Z}^d$ is a point such that

$$y \in \tilde{C}_x^l.$$

We also denote a deformed diamond by $R_{x,y}$ (and it will always be clear in the text whether $R_{x,y}$ is a diamond or deformed diamond).

Definition 4. *It will be useful to write each (exact or deformed) diamond as a non disjoint union of two sets*

$$R_{x,y}^x = \{z \in R_{x,y}, |z - x| \leq f_x |y - x|\}, \quad R_{x,y}^y = \{z \in R_{x,y}, |z - y| \leq f_y |y - x|\},$$

where the pair (f_x, f_y) is such that $1/5 \leq f_x \leq 1$, $1/5 \leq f_y \leq 1$ and $f_x + f_y \geq 1 + \frac{1}{5}$.

This condition on (f_x, f_y) , and the fact that the maximal angle in deformed diamonds is $\pi/16$, indeed ensures that $R_{x,y} = R_{x,y}^x \cup R_{x,y}^y$. Note that the choice of (f_x, f_y) is not unique.

2.3. Estimates on effective conductances.

Definition 5. *Given a diamond or deformed diamond $R_{xy} \subseteq \Lambda$, let D_{xy}^N be the effective resistance of the electrical network with the same conductances $c_{i,j}^{x,y}$ in R_{xy} as in Definition 1, and Neumann boundary conditions on ∂R_{xy} .*

Definition 6. *Fix $b > 1$ and $\alpha \geq 0$. Given $i, j \in \Lambda$, let*

$$\chi_{ij} = \mathbb{1}_{\{B_{ij} \leq b|i-j|^\alpha\}}.$$

Given a deformed diamond $R_{xy} \subseteq \Lambda$ and the two corresponding regions R_{xy}^x and R_{xy}^y , let

$$\bar{\chi}_{xy} = \prod_{j \in R_{xy}^x} \chi_{xj} \prod_{j \in R_{xy}^y} \chi_{yj}.$$

Here is the main proposition of the section. It ensure that under the condition $\bar{\chi}_{xy}$, there is a uniform bound on the effective resistance $D_{x,y}^N$.

Proposition 3. *Fix $\alpha \in (0, 1/4)$ and $b > 0$. There exists a constant $C = \text{Cst}(d, \alpha, b)$ such that for any deformed diamond $R_{x,y} \subseteq \Lambda$, if $\bar{\chi}_{xy}$ is satisfied, then*

$$D_{x,y}^N \leq C/a,$$

where $a = \inf(a_{i,j})$.

Remark 3. *Note that the constant C is independent of the precise shape of the deformed diamond.*

Proposition 3 is proved in Section 6. It partly relies on the following two Lemmas 2 and 3, which will also be useful in other parts of the proof.

Lemma 2 (lemma 2 of [7]). *For all $x, y, z \in \Lambda$,*

$$B_{xz} \leq 2B_{xy}B_{yz}.$$

Proof. Elementary computation. □

Lemma 3. *For all $i, j \in \mathbb{Z}^d$, $i \sim j$, $x, y \in \mathbb{Z}^d$, we have*

$$(2.3) \quad (c_{ij}^{xy}/a)^{-1} \leq 16(B_{iz})^2(B_{jz})^2.$$

for both $z = x$ and $z = y$.

In particular, let $c = b^{-4}/64$, $\beta = 4\alpha$. Assume that $\bar{\chi}_{xy}$ holds, then the electrical network with conductances

$$(\gamma_{i,j})_{i,j \in \Lambda, i \sim j} = (c_{ij}^{xy}/a)_{i,j \in \Lambda, i \sim j},$$

satisfies

$$\gamma_{i,j} \geq c|i - z|^{-\beta}$$

for all $i, j \in R_{xy}^z$, $i \sim j$, for $z = x, y, \cdot$.

The proof of Lemma 3 partly relies on the following two Lemmas, which will also be useful in other parts of the proof.

Proof of Lemma 3. First note that

$$e^{\pm(u_x - u_y)} B_{xy} \geq e^{\pm(u_x - u_y)} \cosh(u_x - u_y) \geq 1/2,$$

so that

$$(2.4) \quad e^{u_i + u_j - u_x - u_y} B_{x,y} \geq \frac{1}{2} \max[e^{u_i + u_j - 2u_x}, e^{u_i + u_j - 2u_y}].$$

Now

$$(2.5) \quad e^{u_i + u_j - 2u_x} = e^{u_i - u_x} e^{u_j - u_x} \geq \frac{1}{4} (\cosh(u_i - u_x) \cosh(u_j - u_x))^{-1} \geq \frac{1}{4} (B_{ix} B_{jx})^{-1}.$$

On the other hand, note that

$$(2.6) \quad B_{ij} \leq 2B_{ix} B_{jx}.$$

Inequalities (2.4)–(2.6) together yield the first part. The second part is a consequence of using $B_{ix} \leq b|i - x|^\alpha$ and $B_{jx} \leq b|j - x|^\alpha \leq 2b|i - x|^\alpha$. \square

2.4. Protected Ward estimates. Next, we obtain a “protected” Ward estimate, as follows.

Lemma 4. *Let $\alpha \geq 0$ and $b > 1$. For all $i = 1, \dots, n$, let $m_i \leq a/4$ and let $R_{x_i y_i}$ be regions whose interiors are disjoint. Then*

$$(2.7) \quad \left\langle \prod_{i=1}^n B_{x_i y_i}^{m_i} (1 - m_i D_{x_i y_i}^N) \right\rangle \leq 1,$$

and

$$(2.8) \quad \left\langle \prod_{i=1}^n B_{x_i y_i}^{m_i} \bar{\chi}_{x_i y_i} (1 - m_i D_{x_i y_i}^N) \right\rangle \leq 1.$$

Inequality (2.8) implies, by Lemma 3, that if, additionally, the regions $R_{x_i y_i}$, $i = 1, \dots, n$ are deformed diamonds and $m_i < a/C$ for all i , then

$$(2.9) \quad \left\langle \prod_{j=1}^n B_{x_j y_j}^{m_j} \bar{\chi}_{x_j y_j} \right\rangle \leq \prod_{j=1}^n (1 - m_j C/a),$$

where C is the constant considered in Proposition 3.

Lemma 1 is proved in Section 5. The rest of the proof is similar to the argument in [7], which consists in deducing upper bounds of $\langle \prod_{j=1}^n B_{x_j y_j}^m \rangle$ for some fixed m from these “protected” estimates, through an induction on the maximal length $|x_i - y_i|$ on the protected estimates, $i = 1 \dots n$, using Chebyshev inequalities in order to deal with the “unprotected” parts of the estimates. We summarise the argument in Section 4.

3. ESTIMATES ON GOOD POINTS

Definition 7. A point $x \in \Lambda$ is called n -good if

$$B_{x,y} \leq b|x-y|^\alpha,$$

for all $y \in \Lambda$ with distance $1 \leq |x-y| \leq 4^n$ from x .

Given $z \in \Lambda$, let $R_n(z)$ (denoted by R_n when there is no ambiguity) the cube with side 4^n and barycenter z . We denote by $\chi_{R_n}^c$ the indicator of the event that there is no n -good point in R_n .

In this section we present an estimate on the event $\chi_{R_n}^c$, more precisely we bound from above the indicator function by a sum of terms involving $\frac{B_{i,j}}{b|j-i|^\alpha}$ for points i, j at distance at most 4^n . This will be used in the main inductive argument in Section 4. It follows lemma 9 of [7], but, for convenience of the reader, we give the proof which is only a few lines long.

If R_n is a hypercube of side 4^n then it is the disjoint union of 4^d sub-hypercubes of side 4^{n-1} . We can select the 2^d corner subcubes that we denote $(R_n^i, i = 1, \dots, 2^d)$ so that $d(R_n^i, R_n^j) > 2 \times 4^{n-1}$ for $i \neq j$. Repeating this procedure hierarchically we can construct a family of cubes (R_n^v) with side 4^{n-k} for v running on the set $\{1, \dots, 2^d\}^k$. We consider the natural structure of rooted $(2^d + 1)$ -regular tree of the set

$$\mathcal{R} = \{\text{root}\} \cup \left(\bigcup_{k=1}^n \{1, \dots, 2^d\}^k \right),$$

where "root" is the root of the tree, corresponding the cube $R_n (= R_n^{\text{root}})$. We denote by d_v the depth of a point $v \in \mathcal{R}$ with $d_{\text{root}} = 0$ and $d_v = k$ if $v \in \{1, \dots, 2^d\}^k$. We denote by \mathcal{T}_n the set of connected subtrees T of \mathcal{R} containing the root and which have the property that any vertex $x \in T$ have either $2d$ or 0 descendant in T . We denote by L_T the set of leaves of such a tree $T \in \mathcal{T}_n$. (Remark that the set $\{L_T, T \in \mathcal{T}_n\}$ is also the set of maximal totally unordered subsets of \mathcal{R} for the natural "genealogical order" on \mathcal{R} .)

For an element $v \in \mathcal{R}$ with $k = d_v$ we set

$$S_{R_n^v}^c = \sum_{\substack{x \in R_n^v, y \in \Lambda \\ 4^{n-k-1} < |x-y| < 4^{n-k}}} \chi_{x,y}^c \leq \sum_{\substack{x \in R_n^v, y \in \Lambda \\ 4^{n-k-1} < |x-y| < 4^{n-k}}} \frac{B_{x,y}^m}{b^m |x-y|^{\alpha m}}$$

Lemma 5. (lemma 9 of [7]) With the notations above we have

$$\chi_{R_n}^c \leq \sum_{T \in \mathcal{T}_n} \prod_{v \in L_T} S_{R_n^v}^c.$$

Proof. If there is no n -good point in R_n then either there is no $(n-1)$ -good point in any of the subcubes $\{R_n^i, i = 1, \dots, 2^d\}$ or else there exists at least one pair $(x, y) \in R_n \times \Lambda$ with $4^{n-1} < |x-y| < 4^n$ and $B_{x,y} > b|x-y|^\alpha$. This gives the first level inequality

$$\chi_{R_n}^c \leq S_{R_n}^c + \prod_{i=1}^{2d} \chi_{R_{n-1}^i}^c.$$

Then the proof follows by induction on the integer n . For $n = 0$, $R_0(z)$ is the singleton z and $\chi_{R_0}^c = S_{R_0}^c = 0$ which initializes the induction. If lemma 5 is valid at level $n-1$, then, obviously, the previous inequality implies it at level n . \square

4. INDUCTIVE ARGUMENT

Fix $b > 1$ and $\alpha \in (0, 1/4)$, which will be chosen later; let $C = \text{Cst}(d, \alpha, b)$ be the constant considered in Proposition 3.

Definition 8. *The sets R_{xy} in our induction are classified as follows:*

- *Class 1: diamonds R_{xy} , with $|x - y| > a^{1/4}$;*
- *Class 2: deformed diamonds R_{xy} , with $|x - y| > a^{1/4}$;*
- *Class 3: deformed diamonds R_{xy} , with $|x - y| \leq a^{1/4}$.*

Our goal in this section is to prove the following theorem.

Theorem 5. *Let $m = a^{1/8}$, let $\rho = 1/2$ and assume $a \geq a_0$ for some constant $a_0 \geq 1$. For all n_1, n_2 and $n_3 \geq 0$, let $R_{x_i y_i}$, $i = 1, \dots, n_1$, $R_{p_j q_j}$, $j = 1, \dots, n_2$, and $R_{r_k s_k}$, $k = 1, \dots, n_3$ be respectively subsets of class 1, 2 and 3. Then we have*

$$\left\langle \prod_{i=1}^{n_1} B_{x_i y_i}^m \prod_{j=1}^{n_2} B_{p_j q_j}^{3m} \bar{\chi}_{p_j q_j} \prod_{k=1}^{n_3} B_{r_k s_k}^{3m} \right\rangle \leq 2^{n_1} (1 + \rho)^{n_2} 2^{n_3}.$$

The proof is by induction on $\max_{1 \leq i \leq n_1} |x_i - y_i|$. Let $(\mathbf{H})_\ell$ be the following statement: Theorem 5 holds if

$$\max_{1 \leq i \leq n_1} |x_i - y_i| \leq \ell.$$

The first step in the induction is the case $n_1 = 0$, proved in Section 4.1.

Let us now show that $(\mathbf{H})_{\ell-1}$ implies $(\mathbf{H})_\ell$. We do the proof only in the case where $n_1 = 1$, the general case being only notationally more involved.

Assume $(\mathbf{H})_{\ell-1}$. Let $x, y \in \Lambda$ be such that $|x - y| = \ell$. Let \tilde{R}_{xy} be the deformed diamond between x and y introduced in Definition 3, with $l = y - x$; let \tilde{R}_{xy}^x and \tilde{R}_{xy}^y be its two parts in Definition 4, respectively from x and y , with $f_x = f_y = 1/(2 \cos(\pi/16))$.

Define

$$u_{xy} = \prod_{j \in \tilde{R}_{xy}^x} \chi_{xj} \prod_{j \in \tilde{R}_{xy}^y} \chi_{yj},$$

and let

$$\mathcal{R}(x, y) = \sum_{z \in \tilde{R}_{xy}^x} B_{xy}^m \chi_{xz}^c \prod_{j: |j-x| < |z-x|} \chi_{xj}.$$

Then it follows from the expansion of the partition of the unity

$$1 = \prod_{j \in \tilde{R}_{xy}^x} (\chi_{xj} + \chi_{xj}^c) \prod_{j \in \tilde{R}_{xy}^y} (\chi_{yj} + \chi_{yj}^c)$$

that

$$(4.1) \quad \langle B_{xy}^m \rangle \leq \langle B_{xy}^m u_{xy} \rangle + \langle \mathcal{R}(x, y) \rangle + \langle \mathcal{R}(y, x) \rangle.$$

The first term in the right-hand side of (4.1) can be upper bounded, by (2.9) in Lemma 4: if $a \geq \text{Cst}(C)$ (recall $m = a^{1/8}$), then

$$(4.2) \quad \langle B_{xy}^m u_{xy} \rangle \leq (1 - mC/a)^{-1} \leq 1 + \rho \leq 3/2.$$

It remains to upper bound $\langle \mathcal{R}(x, y) \rangle$.

Now, $\mathcal{R}(x, y)$ being an expansion over “bad” points z (i.e. sites z such that χ_{xz}^c holds), we expand it into four terms.

(i) First, the sites z close x , i.e. with $|z - x| \leq a^{1/4}$:

$$\mathcal{R}_1(x, y) = \sum_{z \in \tilde{R}_{xy}^x: |z-x| \leq a^{1/4}} B_{xy}^m \chi_{xz}^c \prod_{j: |j-x| < |z-x|} \chi_{xj}.$$

We prove in Section 4.2 that $\langle \mathcal{R}_1(x, y) \rangle \leq 1/16$ if $b \geq \text{Cst}(a)$ and $\alpha m \geq d$ (Case 1 in [7]).

(ii) Second, fix a constant M , which will only depend on the dimension d . For a site z far from x (i.e. with $|z - x| > a^{1/4}$), let

$$v_{x,y,z} = \prod_{\substack{j,k \in \tilde{R}_{xy}^x \cup \tilde{R}_{xy}^y, |j-z| \leq |z-x|^{1/2}, \\ M|z-x|^{1/2} \leq |j-k| \leq |z-x|/5}} \chi_{jk},$$

and let

$$\mathcal{R}_2(x, y) = \sum_{z \in \tilde{R}_{xy}^x: |z-x| > a^{1/4}} B_{xy}^m \chi_{xz}^c v_{x,y,z}^c \prod_{j: |j-x| < |z-x|} \chi_{xj}.$$

Then $v_{x,y,z}^c = 1$ if there is a large scale “bad” event originating from a point near z . We will show in Section 4.3 that the corresponding term $\langle \mathcal{R}_2(x, y) \rangle \leq 1/16$ if $b \geq \text{Cst}(a)$ and $\alpha m \geq 10d$ (Case 2a in [7]).

(iii) Third, we consider the case where $v_{x,y,z}$ holds, i.e. there is no large scale “bad” event near z and, furthermore, there exists a point g with $|g - z| \leq |z - x|^{1/2}$ that is good up to scale $|z - x|^{1/2}$.

More precisely, given $i \in \Lambda$, $R > 0$, let

$$G(i, R) = \prod_{h: |i-h| \leq R} \chi_{ih};$$

then $G(i, R) = 1$ iff i is “good” up to distance R . Recall the similar Definition 7 that a site $x \in \Lambda$ is called n -good if $\chi_{xy} = 1$, for all y with $|y - x| \leq 4^n$.

Let

$$g_{x,y,z} = \max_{g: |g-z| \leq |z-x|^{1/2}} G(g, M|z-x|^{1/2}).$$

Then $g_{x,y,z} = 1$ iff we can find a site g in the ball of radius $|z - x|^{1/2}$ centered at z , which is good up to distance $M|z - x|^{1/2}$.

Now, if $v_{x,y,z} = g_{x,y,z} = 1$, then we can find a deformed diamond from x to g close to z such that $\bar{\chi}_{x,g} = 1$, so that we can apply the induction hypothesis. If we let

$$\mathcal{R}_3(x, y) = \sum_{z \in \tilde{R}_{xy}^x: |z-x| > a^{1/4}} B_{xy}^m \chi_{xz}^c v_{x,y,z} g_{x,y,z} \prod_{j: |j-x| < |z-x|} \chi_{xj},$$

then $\langle \mathcal{R}_3(x, y) \rangle \leq 1/16$ if $a \geq \text{Cst}$ and $\alpha m \geq 3d$ (Case 2b in [7]): this is proved in Section 4.4.

(iv) Fourth, if $g_{x,y,z} = 0$ and $M \leq \text{Cst}(d)$ then there is no good point up to distance $M|z - x|^{1/2}$ in the hypercube of side length $4M|z - x|^{1/2}$. This implies that $\chi_{R_{n(x,z)}(z)}^c$ holds,

where $n(x, z)$ is the part of $M \log |z - x| / (2 \log 4)$. Then, if we let

$$\mathcal{R}_4(x, y) = \sum_{z \in \tilde{R}_{xy}^x: |z-x| > a^{1/4}} B_{xy}^m \chi_{xz}^c v_{x,y,z} \chi_{R_{n(x,z)}(z)}^c \prod_{j: |j-x| < |z-x|} \chi_{xj},$$

we apply the “good points” expansion obtained in Section 3 to show in Section 4.5 that $\langle \mathcal{R}_4(x, y) \rangle \leq 1/16$ if $a \geq \text{Cst}$, $b \geq \text{Cst}(\alpha, d)$ and $\alpha m \geq 4d$ (Case 2c in [7]).

In summary,

$$\langle \mathcal{R}(x, y) \rangle \leq \sum_{i=1}^4 \langle \mathcal{R}_i(x, y) \rangle,$$

and $\langle \mathcal{R}_i(x, y) \rangle \leq 1/16$ is proved in Sections 4.2, 4.3, 4.4 and 4.5, respectively in the cases $i = 1, 2, 3$ and 4.

4.1. Proof of Theorem 5 in the case $n_1 = 0$. The proof is similar to the one of lemma 8 in [7]. First, note that the case $n_3 = 0$ follows from Lemma 4, for $a \geq \text{Cst}(C)$.

Let us do the proof in the case $n_2 = 0$ and $n_3 = 1$, the argument for the general case being only notationally more involved.

Let $\delta > 0$. For all $p, q \in \Lambda$, $p \sim q$, let

$$\xi_{pq} = \mathbb{1}_{\{B_{pq} \leq 1 + \delta\}}.$$

Using $1 \leq \prod_{\{p,q\}, p,q \in R_{xy}} \xi_{pq} + \sum_{\{p,q\}, p,q \in R_{xy}} \xi_{pq}^c$, we have

$$(4.3) \quad \langle B_{xy}^{3m} \rangle \leq \langle B_{xy}^{3m} \prod_{\{p,q\}, p,q \in R_{xy}} \xi_{pq} \rangle + \sum_{\{p,q\}, p,q \in R_{xy}} \langle B_{xy}^{3m} \xi_{pq}^c \rangle.$$

Let us first deal with the first term in the right-hand side of (4.3): $B_{pq} \leq 1 + \delta$ implies

$$0 \leq (U_p - U_q)^2 / 2 \leq \cosh(U_p - U_q) - 1 \leq \delta.$$

Choose $\delta > 0$ such that $a^{1/4} \sqrt{2\delta} = 1$, i.e. $\delta = a^{-1/2} / 2$.

Let $z = x, y$, and assume $\prod_{\{p,q\}, p,q \in R_{xy}} \xi_{pq}$ holds. Then, for all $j \in R_{xy}$,

$$(4.4) \quad |U_z - U_j| \leq a^{1/4} \sqrt{2\delta} = 1.$$

Subsequently, for all $p, q \in R_{xy}$, $p \sim q$,

$$\frac{e^{-2}}{2} e^{2U_z} (S_p - S_q)^2 \leq \frac{e^{U_p + U_q}}{2} (S_p - S_q)^2 \leq B_{pq} - 1 \leq \delta,$$

which implies $e^{U_z} |S_p - S_q| \leq \sqrt{2\delta} e$. Using again our choice of δ , we deduce that, again if $j \in R_{xy}$, $e^{U_z} |S_z - S_j| \leq a^{1/4} \sqrt{2\delta} e = e$, so that

$$(4.5) \quad \frac{e^{U_j + U_z}}{2} (S_j - S_z)^2 \leq \frac{e}{2} e^{2U_z} (S_j - S_z)^2 \leq \frac{e^3}{2}.$$

Inequalities (4.4)-(4.5) together imply that $B_{zj} \leq \cosh(1) + e^3/2 = \text{Cst}$.

Therefore $\bar{\chi}_{xy}$ holds with $b = \text{Cst}$ and $\alpha = 0$ implies

$$(4.6) \quad \langle B_{xy}^{3m} \prod_{\{p,q\}, p,q \in R_{xy}} \xi_{pq} \rangle \leq (1 - 3mC/a)^{-1} \leq 3/2,$$

assuming $a \geq \text{Cst}(C)$ (recall $m = a^{1/8}$).

Let us now deal with the second term in the right-hand side of (4.3): fix $p, q \in R_{xy}$, $p \sim q$, and use that, by Markov inequality,

$$\xi_{pq} \leq \left(\frac{B_{pq}}{1+a} \right)^{a/2}.$$

Let (x_0, \dots, x_n) be a path of minimal distance from x to y inside R_{xy} , which does not go through the edge $\{p, q\}$. By repeated application of Lemma 2,

$$2B_{xy} \leq \prod_{0 \leq j \leq n-1} 2B_{x_j x_{j+1}}.$$

Therefore, letting $\ell = |x - y|$, we have

$$\begin{aligned} \langle B_{xy}^{3m} \xi_{pq}^c \rangle &\leq \frac{2^{3m(\ell-1)}}{(1+\delta)^{a/2}} \langle B_{pq}^{a/2} \prod_{0 \leq j \leq n-1} B_{x_j x_{j+1}}^{3m} \rangle \leq \frac{2^{3m(\ell-1)}}{(1+\delta)^{a/2}} 2 \left(1 - \frac{3m}{a} \right)^{-\ell} \\ (4.7) \quad &\leq \exp(3m\ell - a\delta/3) \leq \exp(3a^{3/8} - a^{1/2}/6). \end{aligned}$$

In the second inequality, we use (2.7) and note that, if $r \sim s$, then $D_{rs}^N = a_{rs} \geq a$. In the third inequality we assume $a \geq \text{Cst}$ and, for $x \in [0, 1/2]$, $1-x \leq e^{-x}$ and $(1+x)^{-1} \leq e^{-2x/3}$. Finally, we use $\delta = a^{-1/2}/2$, $m = a^{1/8}$ and $\ell \leq a^{1/4}$ (since R_{xy} is of Class 3) in the last inequality.

In summary, (4.3), (4.6) and (4.7) together imply, if $a \geq \text{Cst}(C)$,

$$\langle B_{xy}^{3m} \rangle \leq 3/2 + \text{Cst}(d)\ell^d \exp(3a^{3/8} - a^{1/2}/6) \leq 2$$

if $a \geq \text{Cst}(d)$.

4.2. Proof of $\langle \mathcal{R}_1(x, y) \rangle \leq 1/16$. Let $z \in \tilde{R}_{xy}^x$ be such that $|z - x| \leq a^{1/4}$. Using $\chi_{xz}^c \leq B_{xz}^{2m} b^{-2m} |z - x|^{-2\alpha m}$ and Lemma 2, we obtain

$$B_{xy}^m \chi_{xz}^c \leq 2^m B_{xz}^m B_{zy}^m \chi_{xz}^c \leq 2^m b^{-2m} |z - x|^{-2\alpha m} B_{xz}^{3m} B_{zy}^m.$$

In order to apply the induction assumption, we would need to construct a deformed diamond R_{xz} and diamond R_{zy} which do not intersect within R_{xy} . This is not true in general, but we can add an intermediate point $a \in R_{xy}$ such that R_{xz} , R_{za} and R_{ay} are respectively one deformed diamond and two diamonds within R_{xy} and disjoint, except at endpoints (see Figure 1, and Lemma 12 [1] in [7] for more details). Now, using again Lemma 2,

$$B_{xy}^m \chi_{xz}^c \leq 2^{2m} b^{-2m} |z - x|^{-2\alpha m} B_{xz}^{3m} B_{za}^m B_{ay}^m.$$

Therefore, using the induction assumption,

$$\langle \mathcal{R}_1(x, y) \rangle \leq \sum_{z \in \tilde{R}_{xy}^x : |z-x| \leq a^{1/4}} 4^m b^{-2m} |z - x|^{-2\alpha m} 4(1+\rho) \leq \left(\frac{4}{b^2} \right)^m \text{Cst} \sum_{r=1}^{a^{1/4}} r^{d-1-2\alpha m} \leq 1/16$$

if $b \geq \text{Cst}(a)$ and $\alpha m \geq d$.

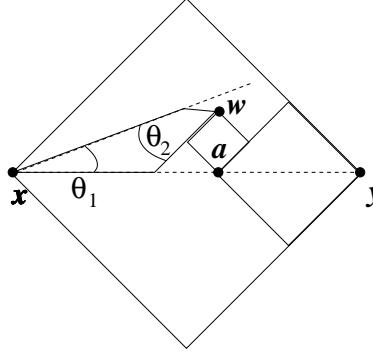


FIGURE 1. (Figure 5 in [7]) We add one intermediate point a . The two angles θ_1 and θ_2 are greater than $\pi/8$.

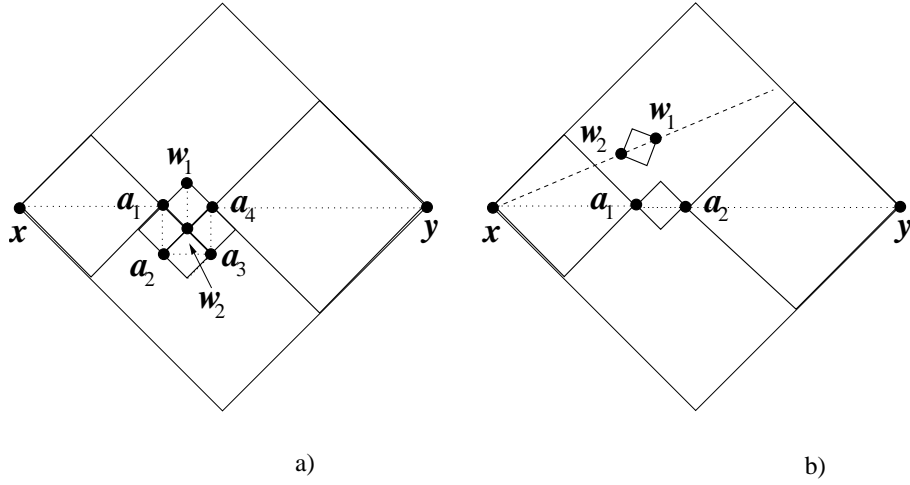


FIGURE 2. (Figure 6 in [7]) a) If the pair $w_1 w_2$ is right in the middle, then we need to add four intermediate points a_1, \dots, a_4 in order to find a minimal connected path around $w_1 w_2$ paved with disjoint diamonds. b) Even if the pair $w_1 w_2$ is located on the boundary of \tilde{R}_{xy}^x , the region $R_{w_1 w_2}$ still lies inside R_{xy} .

4.3. **Proof of $\langle \mathcal{R}_2(x, y) \rangle \leq 1/16$.** Let $z \in \tilde{R}_{xy}^x$ be such that $|z - x| > a^{1/4}$, and let $j, k \in \tilde{R}_{xy}^x \cup \tilde{R}_{xy}^y$ such that

$$(4.8) \quad |j - z| \leq |z - x|^{1/2} \text{ and } M|z - x|^{1/2} \leq |j - k| \leq |z - x|/5.$$

As above, we use

$$(4.9) \quad \chi_{jk}^c \leq B_{jk}^m b^{-m} |j - k|^{-\alpha m}.$$

In order to apply the induction assumption, we need to expand $B_{jk}^{2m} B_{xy}^m$ into a product of terms arising from disjoint diamonds within R_{xy} . It is an easy geometric result to show that, under our assumptions on z, j and k , we can choose four intermediate points $a_i \in R_{xy}$, so that $R_{xa_1}, R_{a_i a_{i+1}}$ ($i = 1, \dots, 3$) and $R_{a_4 y}$ are diamonds with disjoint interiors which do not overlap with the diamond R_{jk} (see Figure 2, and Lemma 12 [2] in [7] for more details). Now, using (4.9) and Lemma 2,

$$B_{xy}^m \chi_{jk}^c \leq 2^{4m} B_{xa_1}^m \prod_{i=1}^3 B_{a_i a_{i+1}}^m B_{a_4 y}^m B_{jk}^m b^{-m} |j-k|^{-\alpha m},$$

which implies, using the induction assumption,

$$\langle B_{xy}^m \chi_{jk}^c \rangle \leq 2^{4m} 2^6 b^{-m} |z-x|^{-\alpha m/2}.$$

Now, for each $z \in \tilde{R}_{xy}^x$, there are of the order of $|z-x|^{d+d/2}$ pairs (j, k) satisfying (4.8). Therefore

$$\langle \mathcal{R}_2(x, y) \rangle \leq \left(\frac{16}{b} \right)^m \text{Cst} \sum_{r > a^{1/4}} r^{(d-1)+d+d/2-\alpha m/2} \leq \frac{1}{16}$$

if $b \geq \text{Cst}(a)$ and $\alpha m \geq 10d$.

4.4. Proof of $\langle \mathcal{R}_3(x, y) \rangle \leq 1/16$. Let $z \in \tilde{R}_{xy}^x$ be such that $|z-x| > a^{1/4}$.

If $v_{x,y,z} = g_{x,y,z} = 1$, then there exists g with $|g-z| \leq |z-x|^{1/2}$ such that, for all $h \in R_{xy}$ with $|g-h| \leq |z-x|/5$, χ_{gh} holds. Let R_{xg} be the deformed diamond in Definition 3, with $l = y-x$, and choose $f_x = (|z-x|-1)/|g-x| = 1 - O(a^{-1/8})$, $f_y = 1/5$ in Definition 4. Then $\bar{\chi}_{xg}$ holds.

On the other hand, if χ_{xz}^c occurs then, using $B_{gz} \leq b|z-g|^\alpha \leq b|z-x|^{\alpha/2}$ and Lemma 2, we deduce that $2B_{xg} \geq B_{xz}/B_{gz} \geq b|z-x|^\alpha/(b|z-x|^{\alpha/2}) = |z-x|^{\alpha/2}$.

Hence

$$B_{xy}^m \chi_{xz}^c \bar{\chi}_{xg} \leq 2^m B_{xg}^m B_{gy}^m \chi_{xz}^c \bar{\chi}_{xg} \leq 2^{3m} (B_{xg}^{3m} \bar{\chi}_{xg}) B_{gy}^m |z-x|^{-\alpha m}.$$

As in the proof of the case $i = 1$, we introduce an intermediate point $a \in R_{xy}$ such that R_{ga} and R_{ay} are diamonds, disjoint from each other and from R_{xg} , except at endpoints (see Figure 1, and Lemma 12 [1] in [7] for more details).

Therefore

$$\langle B_{xy}^m \chi_{xz}^c \bar{\chi}_{xg} \rangle \leq 2^{3m} (1 + \rho) 2^2 |z-x|^{-\alpha m}.$$

There are less than $|z-x|^{d/2}$ choices for g , so that

$$\langle \mathcal{R}_3(x, y) \rangle \leq 2^{3m} \sum_{r > a^{1/4}} r^{d-1+d/2-\alpha m} \leq 1/16$$

if $\alpha m \geq 3d$ and $a \geq \text{Cst}$.

4.5. Proof of $\langle \mathcal{R}_4(x, y) \rangle \leq 1/16$. If $a \geq \text{Cst}$, then $R_{n(x,z)}(z)$ is inside R_{xy} ; recall that its side length is of the order of $|z-x|^{1/2} \ll |y-x|$. As in the proof of the case $i = 2$, we can choose four intermediate points $a_i \in R_{xy}$, so that R_{xa_1} , $R_{a_i a_{i+1}}$ ($i = 1, \dots, 3$) and $R_{a_4 y}$ are diamonds with disjoint interiors which do not overlap with the hypercube $R_{n(x,z)}(z)$ (see Figure 2, and Lemma 12 [2] in [7] for more details).

Now, using Lemma 5,

$$B_{xy}^m \chi_{R_{n(x,z)}(z)}^c \leq 2^{4m} B_{xa_1}^m \prod_{i=1}^3 B_{a_i a_{i+1}}^m B_{a_4 y}^m \sum_{T \in \mathcal{T}_n} \sum_{\substack{x_v \in R_v, y_v \in \Lambda, v \in L_T \\ 4^{n-k_v-1} < |x_v - y_v| \leq 4^{n-k_v}}} \prod_{v \in L_T} \frac{B_{x_v y_v}^m}{b^m |x_v - y_v|^{\alpha m}}.$$

This implies, letting $n_v = n - d_v$ and using the induction assumption,

$$\langle B_{xy}^m \chi_{R_{n(x,z)}^c}(z) \rangle \leq 2^{4m} 2^6 \sum_{T \in \mathcal{T}_n} \prod_{v \in L_T} \frac{(4^{n_v})^d (4^{n_v} 2)^d}{b^m 4^{(n_v-1)\alpha m}} = 2^{4m+6} I_{n(x,z)}$$

where, for all $n \geq 0$,

$$I_n = \sum_{T \in \mathcal{T}_n} \prod_{v \in L_T} \gamma 2^{-\zeta n_v},$$

with $\gamma = (4^\alpha/b)^m 2^d$, $\zeta = 2(\alpha m - 2d)$.

It follows from the structure of the trees \mathcal{T}_n that

$$I_n = \gamma 2^{-\zeta n} + (I_{n-1})^{2^d}, \quad I_0 = \gamma.$$

Assume $\alpha m \geq 4d$ and $b \geq \text{Cst}(\alpha, d)$, so that $\gamma \leq 1/4$ and $\zeta \geq \alpha m$. Then we deduce by elementary induction that

$$I_n \leq 2^{-\alpha m n} \text{ for all } n \geq 1.$$

Note that $4^{n(x,z)} \geq 2^M \sqrt{|z-x|}$. In summary,

$$\langle \mathcal{R}_4(x, y) \rangle \leq 2^{4m} 2^{-\alpha m M/2} \text{Cst} \sum_{r > a^{1/4}} r^{d-1-\alpha m/4} \leq 1/16$$

if $\alpha m \geq 8d$ and $a \geq \text{Cst}$.

5. PROOF OF WARD INEQUALITIES: LEMMAS 1 AND 4

We start this section with two lemmas, that will be useful in the subsequent proofs. The first is an elementary lemma which expresses the equivalent resistance on a conductance network as a quadratic form and relates the quantity $D_{x,y}$ to the corresponding term in [7].

Lemma 6. *Let (Λ, E) be a finite connected graph and $(c_e)_{e \in E}$ a conductance network on E . We set $c_i = \sum_{j \sim i} c_{i,j}$. Let $i_0 \in \Lambda$ be a fixed vertex and M be the matrix given by*

$$(M_{i,j}) = \begin{cases} c_{i,j}, & i \neq j \\ -c_i, & i = j \end{cases}$$

(which is the Matrix of the generator of the Markov process with jump rates $(c_{i,j})$). Let N be the restriction of M to $\Lambda \setminus \{i_0\}$. Denote by G the $\Lambda \times \Lambda$ symmetric matrix defined by $G(i_0, y) = G(y, i_0) = 0$ for any y and $G(x, y) = -N_{x,y}^{-1}$ if $x, y \neq i_0$. If $D_{x,y}$ is the equivalent resistance between x and y , then

$$D_{x,y} = G(x, x) - 2G(x, y) + G(y, y) = \langle (\delta_x - \delta_y), G(\delta_x - \delta_y) \rangle.$$

Remark 4. *In comparison with [7], it means that the term $G_{x,y}$ which appears in formula (5.4) in [7] is the equivalent resistance between x and y with conductances $c_{i,j}^{x,y} = e^{-t_x - t_y} B_{x,y} \beta_{i,j} e^{t_i + t_j}$.*

Proof. We first interpret probabilistically the matrix G . Let $\mathcal{G}_{i_0}(x, y)$ be the Markov chain with transition probabilities $p_{i,j} = \frac{c_{i,j}}{c_i}$ and killed at its first entrance hitting time of i_0

$$\mathcal{G}_{i_0}(x, y) = \mathbb{E}_x \left(\sum_{k=0}^{H_{i_0}} \mathbb{1}_{X_k=y} \right),$$

where $H_{i_0} = \inf\{k \geq 0, X_k = i_0\}$. Then, clearly,

$$G(x, y) = \frac{1}{c_x} \mathcal{G}_{i_0}(x, y),$$

Then exercise 2.61 in chapter 2 of [8] yields the result. \square

The next lemma ensures that the joint density $\mu_{a,V}(u, s)$ given in (2.2) has bounded “moments” up to a certain order.

Lemma 7. *Let (x_i, y_i) , $i = 1, \dots, n$ be n pairs of nearest neighbor points and let e_1, \dots, e_n be the corresponding undirected edges in E . Then*

$$\left\langle \prod_{j=1}^n B_{e_j}^{m_j} \right\rangle \leq 2^n$$

for any choice of m_1, \dots, m_n such that $m_j \leq a/2$ for all $j = 1, \dots, n$.

Proof. By expression (2.2), we have

$$\left\langle \prod_{j=1}^n B_{e_j}^{m_j} \right\rangle = \frac{1}{(2\pi)^{(N-1)/2}} \int \left[\prod_e \frac{1}{B_e^{\bar{a}_e}} \right] D[M_{a,V}(u, s)] e^{-\sum_{j \in \Lambda} u_j} \prod_{k \neq i_0} du_k ds_k,$$

where we set $\bar{a}_{e_j} = a_{e_j} - m_j$ for $j = 1, \dots, n$ and $\bar{a}_e = a_e$ for all other edges. Note that $\bar{a}_{e_j} \geq a_{e_j}/2$ since $m_j \leq a/2$ for all j . Expanding the minor as a sum over spanning trees we deduce

$$\begin{aligned} \left\langle \prod_{j=1}^n B_{e_j}^{m_j} \right\rangle &= \frac{1}{(2\pi)^{(N-1)/2}} \sum_T \int \left[\prod_e \frac{1}{B_e^{\bar{a}_e}} \right] \left[\prod_{e \in T} c_e \right] e^{-\sum_{j \in \Lambda} u_j} \prod_{k \neq i_0} du_k ds_k, \\ &\leq 2^n \frac{1}{(2\pi)^{(N-1)/2}} \int \left[\prod_e \frac{1}{B_e^{\bar{a}_e}} \right] D[\tilde{M}_{\bar{a},V}(u, s)] e^{-\sum_{j \in \Lambda} u_j} \prod_{k \neq i_0} du_k ds_k = 2^n \int d\mu_{\bar{a},V}(u, s) = 2^n \end{aligned}$$

where we have used the bound $a_{e_j} \leq 2(a_{e_j} - m_j)$ so we can replace a_e with \bar{a} in the determinant. \square

Proof of lemma 1. For more readability we provide an elementary derivation of the Ward identity which does not involve fermionic integral, even though it could be deduced from the more general proof of lemma 4. Consider the graph (V, \tilde{E}) where we add an extra edge $\tilde{e} = \{x, y\}$ to E (possibly creating a double edge). We put a weight $a_{\tilde{e}} = -m$ on this edge. Denote by $\tilde{\mu}_{a,V}(u, s)$ the corresponding density, and by $\tilde{M}_{a,V}$ the corresponding matrix in (2.1). Using the expression (2.2), we deduce

$$|\tilde{\mu}_{a,V}(u, s)| \leq \left[\prod_{e \in E} \frac{1}{B_e^{a_e}} \right] B_{xy}^m D[\tilde{M}_{|a|,V}] = \left[\prod_{e \in \tilde{E}} \frac{1}{B_e^{a_e}} \right] B_{xy}^{2m} D[\tilde{M}_{|a|,V}]$$

where $|a|_{xy} = m$ and $|a_e| = a_e > 0$ for all $e \in E$. Now let γ a simple path in E connecting x to y . Then, by Lemma 2,

$$B_{xy}^m \leq 2^{m(|\gamma|-1)} \prod_{e \in \gamma} B_e^{2m},$$

and, by the same argument as in the proof Lemma 7 above,

$$|\tilde{\mu}_{a,V}(u, s)| \leq 2^{m(|\gamma|-1)} \left[\prod_{e \in \gamma} B_e^{2m} \right] \tilde{\mu}_{|a|,V}(u, s) \leq 2^{m(|\gamma|-1)} 2^{|\gamma|} \tilde{\mu}_{|\bar{a}|,V}(u, s)$$

where $|\bar{a}|_e = |a|_e - 2m$ for all $e \in \gamma$ and $|\bar{a}|_e = |a|_e$ otherwise. By Proposition 2, $d\tilde{\mu}_{|\bar{a}|,V}(u, s)$ is a probability measure on \tilde{E} . The bound above holds for any $m \leq a/4$, hence $\tilde{\mu}_{a,V}(u, s)$ is integrable. Moreover it is an analytic function in the parameters a_e , and therefore

$$(5.1) \quad \int \tilde{\mu}_{a,V}(u, s) \prod_{k \neq i_0} du_k ds_k = 1.$$

Now let $N(u, s)$ be the restriction of $M_V(u, s)$ to the subset $V \setminus \{i_0\}$, and let $\tilde{N}(u, s)$ be the corresponding matrix for the new graph. Expanding the determinant with respect to the extra term coming from the new edge \tilde{e} we deduce, letting $N = N(u, s)$,

$$\begin{aligned} \det(\tilde{N}(u, s)) &= \det(N) \\ &\quad - (-m) \frac{e^{u_x + u_y}}{B_{x,y}} (\mathbb{1}_{x \neq i_0} \det(N)_{x,x} - 2\mathbb{1}_{x \neq i_0, y \neq i_0} \det(N)_{x,y} + \mathbb{1}_{y \neq i_0} \det(N)_{y,y}) \\ &= \det(N) \left[1 - m \frac{e^{u_x + u_y}}{B_{x,y}} < (\delta_x - \delta_y), G(\delta_x - \delta_y) > \right], \end{aligned}$$

where in the previous expressions, $\det(N(u, s))_{x,y}$ is the minor where we remove line x and column y , and where in the last line G is the matrix defined in Lemma 6, with the conductances $c_{i,j}$ defined in Proposition 2. Using lemma 6, we deduce

$$\det(\tilde{N}(u, s)) = \det(N(u, s))(1 - mD_{x,y}).$$

Therefore

$$1 = \int \tilde{\mu}_{a,V}(u, s) \prod_{k \neq i_0} du_k ds_k = \int B_{x,y}^m (1 - mD_{x,y}) \mu_{a,V}(u, s) \prod_{k \neq i_0} du_k ds_k.$$

□

Proof of Lemma 4. In [7], the protected Ward estimates are a consequence of Berezin identity stated in appendix C, proposition 2 of [7]. The starting point is to write the determinant term $D[M(W, u)]$ as a fermionic integral (cf e.g. [5, 7]) with new pairs of anti commuting variables $(\bar{\psi}_i, \psi_i)$. This leads to

$$(5.2) \quad \mu_{a,V}(u, s, \bar{\psi}, \psi) = \left[\prod_{e \in E} \frac{1}{B_e^{a_e}} \right] e^{-\sum_{e \in E} \frac{a_e}{B_e} (S_e - B_e)} e^{-\sum_{j \in \Lambda} u_j},$$

where

$$S_{i,j} = B_{ij} + e^{u_i + u_j} (\bar{\psi}_i - \bar{\psi}_j)(\psi_i - \psi_j)$$

is the same supersymmetric expression introduced in [7], $u_{i_0} = s_{i_0} = 0$ and $\bar{\psi}_{i_0} = \psi_{i_0} = 0$. Then

$$d\mu_{a,V}(u, s) = \int d\mu_{a,V}(u, s, \bar{\psi}, \psi) \prod_{k \neq i_0} d\bar{\psi}_k d\psi_k.$$

From the mathematical point of view, the fermionic integral should be understood as an interior product with respect to the variables $(\bar{\psi}_k, \psi_k)$, cf e.g. [5]. Since the fermionic variables are antisymmetric, we have $(S_e - B_e)^2 = 0$, and

$$e^{-\frac{a_e}{B_e}(S_e - B_e)} = 1 - \frac{a_e}{B_e}(S_e - B_e) = \frac{1}{1 + \frac{a_e}{B_e}(S_e - B_e)}$$

Then

$$(5.3) \quad \mu_{a,V}(u, s, \bar{\psi}, \psi) = \left[\prod_{e \in E} \frac{1}{S_e^{a_e}} \right] e^{-\sum_{j \in \Lambda} u_j}$$

If $f(s_{ij})$ is a smooth function of the variable $(s_{i,j})$, we understand $f(S_{i,j})$ as the function obtained by Taylor expansion of the fermionic variables $(\bar{\psi}_i, \psi_i, \bar{\psi}_j, \psi_j)$; this expansion is finite since the fermionic variables are antisymmetric. Hence, formally $f(S_{i,j})$ is function with values in the exterior algebra constructed from $(\bar{\psi}_i, \psi_i, \bar{\psi}_j, \psi_j)$. The same can be generalized to $f[(S_{ij})_{i,j \in V \times V}]$. The Berezin identity (see for instance proposition 2 in [7]) implies that for any smooth function $f[(s_{ij})_{ij}]$, then

$$(5.4) \quad \int f[(S_{ij})_{ij}] d\mu_{a,V}(u, s, \bar{\psi}, \psi) = f(0),$$

if $f[(s_{ij})_{ij}]$ is integrable with respect to $d\mu_{a,V}(u, s, \bar{\psi}, \psi)$ (which means that after integration with respect to the fermionic variables it is integrable in the usual sense).

Remark that if f is a polynomial (of degree bounded by a), then it is a direct consequence of the fact that the measure $d\mu_{a,V}(u)$ has integral one, which can be proved either by supersymmetric or probabilistic arguments (as we did in (5.1) above). Indeed let (x_i, y_i) , $i = 1, \dots, n$ be n pairs of points such that $R_{x_i y_i}$ are regions whose interiors are disjoint. As in the proof of lemma 1 above let $\tilde{E} = E \cup (\cup_{i=1}^n \{x_i, y_i\})$ the graph obtained by adding the edges $\{x_i, y_i\}$. We assign to each new edge $\{x_i, y_i\}$ the conductance $a_{x_i, y_i} = -m_i$. Let $\tilde{M}_{a,V}$ be the corresponding matrix. Then

$$\langle \prod_{i=1}^n S_{x_i y_i}^{m_i} \rangle = \frac{1}{(2\pi)^{(N-1)/2}} \int \left[\prod_{e \in \tilde{E}} \frac{1}{B_e^{a_e}} \right] D[\tilde{M}_{a,V}(u, s)] e^{-\sum_{j \in \Lambda} u_j} \prod_{k \neq i_0} du_k ds_k,$$

In order to show that the density is integrable, we proceed as in the proof of Lemma 1. This time we have to choose n simple paths γ_i , $i = 1, \dots, n$ in E connecting x_i to y_i , which do not intersect; this is possible, since the regions $R_{x_i y_i}$ have disjoint interiors. Then

$$\langle \prod_{i=1}^n S_{x_i y_i}^{m_i} \rangle = 1$$

Now by the same arguments as in [7, lemma 7] we have

$$D[\tilde{M}_{a,V}(u, s)] \leq D[M_{a,V}(u, s)] \prod_{j=1}^n (1 - m_j D_{x_j y_j}^N).$$

This completes the proof of (2.7). Finally to prove the “protected” Ward estimate (2.7) we proceed exactly as in [7, lemma 6]. The main idea is to approximate the characteristic function $\chi(x)$ by a sequence of smooth decreasing functions $\chi_\delta(x)$. Now, by symmetry,

$$\langle S_{xy}^m \chi_\delta(S_{xy}) \rangle = 1 \quad \forall x, y.$$

Integration over the fermionic variables yields

$$\begin{aligned} \langle S_{xy}^m \chi_\delta(S_{xy}) \rangle &= \langle B_{xy}^m \chi_\delta(B_{xy}) e^{\left[m + \frac{\chi'(B_{xy})}{\chi(B_{xy})} \right] (S_{xy} - B_{xy})} \rangle \\ &= \frac{1}{(2\pi)^{(N-1)/2}} \int \left[\prod_{e \in E} \frac{1}{B_e^{a_e}} \right] B_{xy}^m \chi_\delta(B_{xy}) D[\tilde{M}_{a,V}(u, s)] e^{-\sum_{j \in \Lambda} u_j} \prod_{k \neq i_0} du_k ds_k, \end{aligned}$$

where $\tilde{M}_{a,V}$ is defined on the graph $\tilde{E} = E \cup (x, y)$ and the edge (x, y) has conductance

$$a_{xy} = -m + B_{xy} \frac{-\chi'(B_{xy})}{\chi(B_{xy})} e^{u_x + u_y} > -m,$$

since $\chi' < 0$. Hence

$$D[\tilde{M}_{a,V}(u, s)] > D[\tilde{M}_{\bar{a},V}(u, s)] = D[M_{a,V}(u, s)](1 - mD_{xy})$$

where $\bar{a}_{xy} = -m$ and $\bar{a}_e = a_e$ for all other e . Finally

$$\langle B_{xy}^m \chi_\delta(B_{xy}) (1 - mD_{xy}) \rangle \leq 1.$$

The proof of the general statement follows from a combination of this argument with the ideas used for the estimate (2.7). \square

6. PROOF OF PROPOSITION 3

Denote by

$$E_{x,y} = \{\{i, j\}, i \in R_{x,y}, j \in R_{x,y}, i \sim j\}$$

the set of non-directed edges in $R_{x,y}$. We denote by $\tilde{E}_{x,y}$ the associated set of **directed** edges. By construction, $aD_{x,y}^N$ is the effective conductance between x and y of the network with edges $E_{x,y}$ and conductances $(\gamma_{i,j})_{\{i,j\} \in E_{x,y}}$. Let $\mathcal{F}_{x,y}$ be the set of unit flows from x to y with support in $\tilde{E}_{x,y}$: precisely, $\theta \in \mathcal{F}_{x,y}$ if θ is a function $\theta : \tilde{E}_{x,y} \rightarrow \mathbb{R}$ which is antisymmetric (i.e. $\theta(i, j) = -\theta(j, i)$) and such that

$$\text{div}(\theta) = \delta_x - \delta_y,$$

where $\text{div} : R_{x,y} \rightarrow \mathbb{R}$ is the function

$$\text{div}(\theta)(i) = \sum_{j \in R_{x,y}, j \sim i} \theta(i, j).$$

Recall that (see for instance [8], Chapter 2)

$$(6.1) \quad aD_{x,y}^N = \inf_{\theta \in \mathcal{F}_{x,y}} \sum_{\{i,j\} \in E_{x,y}} \frac{1}{\gamma_{i,j}} (\theta(i, j))^2.$$

The strategy is now to construct explicitly a flow θ such that under the condition $\bar{\chi}_{x,y}$, the energy (6.1) is bounded by a constant depending only on d , α and b . This flow will be constructed as an integral of flows associated to sufficiently spread paths.

Remind that a deformed diamond is a set of the following form

$$\mathbb{Z}^d \cap (\tilde{C}_x^l \cap \tilde{C}_y^{x-y}),$$

(plus a few points close to x and to y so that the set is connected in \mathbb{Z}^d) where $x \in \mathbb{Z}^d$, $l \in \mathbb{R}^d$, $l \neq 0$ and $y \in \mathbb{Z}^d$ is a point such that $y \in \tilde{C}_x^l$

For $z \in \mathbb{R}^d$ we set

$$r(z) = \frac{(z-x) \cdot (y-x)}{|y-x|^2},$$

and $p(z) = x + r(z)(y-x)$ the projection of z on the line (x, y) .

For $h \in (0, 1)$ we denote

$$\hat{R}_{x,y}^x = \{i \in R_{x,y}, r(i) \leq h\}, \quad \hat{R}_{x,y}^y = \{i \in R_{x,y}, r(i) \geq h\}.$$

From the assumption on f_x, f_y , there exists $h \in [1/10, 9/10]$ such that $\hat{R}_{x,y}^x \subseteq R_{x,y}^x$ and $\hat{R}_{x,y}^y \subseteq R_{x,y}^y$ ¹. We fix now such a $h \in [1/10, 9/10]$. We set

$$\Delta_h = \{z \in \mathbb{R}^d, r(z) = h\} \cap (\tilde{C}_x^l \cap \tilde{C}_y^{x-y}).$$

It is clear from the construction that

$$|\Delta_h| \geq \text{Cst}(d)|x-y|^{d-1},$$

where $|\Delta_h|$ is the surface of Δ_h (note that $\text{Cst}(d)$ does not depend on the value of $h \in [1/10, 9/10]$).

To any path $\sigma = (x_0 = x, \dots, x_n = y)$ from x to y we can associate the unit flow from x to y defined by

$$\theta_\sigma = \sum_{i=1}^n \mathbb{1}_{(x_{i-1}, x_i)} - \mathbb{1}_{(x_i, x_{i+1})}.$$

For $u \in \Delta_h$, let L_u be the union of segments

$$L_u = [x, u] \cup [u, y].$$

Clearly $L_u \subseteq \tilde{C}_x^l \cap \tilde{C}_y^{x-y}$ by convexity. There is a constant $\text{Cst}(d)$, such that for any $u \in \Delta_h$, we can find a simple path σ_u in $R_{x,y}$ from x to y such that for all $k = 0, \dots, |\sigma_u|$

$$(6.2) \quad \text{dist}(L_u, \sigma_u(k)) \leq \text{Cst}(d),$$

and we define θ as

$$\theta = \frac{1}{|\Delta_h|} \int_{\Delta_h} \theta_{\sigma_u} du,$$

which is a unit flow from x to y .

The path σ_u can visit a vertex i only if $\text{dist}(i, L_u) \leq c_0 = \text{Cst}(d)$. This implies that, for all i ,

$$\sum_{j: j \sim i} |\theta(i, j)| \leq \frac{1}{|\Delta_h|} \int_{\Delta_h} \mathbb{1}_{\{\text{dist}(i, L_u) \leq c_0\}} du.$$

Now if $r(i) \in [0, h]$, then

$$\int_{\Delta_h} \mathbb{1}_{\{\text{dist}(i, L_u) \leq c_0\}} du \leq \text{Cst}(d) \left(\frac{h}{r(i)} \right)^{d-1}$$

¹Indeed, if $i \in R_{x,y}$, the angle $\angle(z, i), (x, y) \leq \pi/8$ for $z = x, y$. Hence, if $i \in \hat{R}_{x,y}^x$, then $|i-x| \leq \frac{h}{\cos(\pi/8)}|x-y|$. Hence $\hat{R}_{x,y}^x \subseteq R_{x,y}^x$ as soon as $h \leq f_x \cos(\pi/8)$. But $\cos(\pi/8) \geq 0.92$ and $f_x \geq 1/5$ so that $f_x \cos(\pi/8) \geq 0.18$. Similarly $\hat{R}_{x,y}^y \subseteq R_{x,y}^y$ as soon as $1-h \leq f_y \cos(\pi/8)$, since $f_y \cos(\pi/8) \geq 0.18$. Using $f_x + f_y > 1 + 1/5$ it implies that $(f_x + f_y) \cos(\pi/8) > 1$ and we can find $h \in [1/10, 9/10]$ such that $\hat{R}_{x,y}^x \subseteq R_{x,y}^x$ and $\hat{R}_{x,y}^y \subseteq R_{x,y}^y$.

Hence, if $r(i) \in [0, h]$

$$\sum_{j: j \sim i} |\theta(i, j)| \leq \text{Cst}(d) \frac{1}{|x - y|^{d-1}} \left(\frac{|x - y|}{|i - x|} \right)^{d-1} = \frac{\text{Cst}(d)}{|i - x|^{d-1}}.$$

Similarly we have if $r(i) \in [h, 1]$

$$\sum_{j: j \sim i} |\theta(i, j)| \leq \frac{\text{Cst}(d)}{|i - y|^{d-1}}.$$

Now, under the condition $\bar{\chi}_{x,y}$, we know that if $i \in \hat{R}_{x,y}^x \subseteq R_{x,y}^x$, then $\gamma_{i,j} \geq c|i - x|^{-\beta}$ and if $i \in \hat{R}_{x,y}^y \subseteq R_{x,y}^y$, then $\gamma_{i,j} \geq c|i - y|^{-\beta}$ with $\beta = 4\alpha < 1$. This implies

$$\begin{aligned} aD_{x,y}^N &\leq \sum_{i \in \hat{R}_{x,y}^x, i \neq x} c^{-1}|i - x|^\beta \left(\sum_{j: j \sim i} |\theta(i, j)| \right)^2 + \sum_{i \in \hat{R}_{x,y}^y, i \neq y} c^{-1}|i - y|^\beta \left(\sum_{j: j \sim i} |\theta(i, j)| \right)^2 \\ &\leq \text{Cst}(d) \left(\sum_{i \in \hat{R}_{x,y}^x, i \neq x} c^{-1}|i - x|^{\beta-2(d-1)} + \sum_{i \in \hat{R}_{x,y}^y, i \neq y} c^{-1}|i - y|^{\beta-2(d-1)} \right) \end{aligned}$$

For $k \in \mathbb{N}$,

$$|\{i \in R_{x,y}, k \leq |i - x| < k + 1\}| \leq \text{Cst}(d)k^{d-1},$$

and similarly for x replaced by y . Therefore

$$aD_{x,y}^N \leq \text{Cst}(d) \sum_{k=1}^{\infty} c^{-1}k^{\beta-(d-1)} \leq \text{Cst}(d, \alpha, b).$$

since $\beta = 4\alpha < 1$ and $d \geq 3$.

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